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# CALCULATION OF TEMPERATURE DISTRIBUTIONS IN THIN SHELLS OF REVOLUTION BY THE FINITE-ELEMENT METHOD

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16. Abstract  <p>This report describes a finite-element procedure and the resulting computer program for computing steady-state temperature distributions in thin shells of revolution having general meridional curvature. The procedure is applicable to shells subjected to a very general set of thermal loading conditions including applied heat flux at the surfaces, convective heat transfer with the surroundings, and equivalent loadings due to specified temperatures at points on the shell. The program is used in a number of calculations for configurations of interest and comparisons are made with exact analytical solutions.</p>					
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# CALCULATION OF TEMPERATURE DISTRIBUTIONS IN THIN SHELLS OF REVOLUTION BY THE FINITE-ELEMENT METHOD

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## SUMMARY

This report describes a procedure for computing the steady-state temperature distributions in thin shells of revolution having general meridional curvatures. The procedure is based on the finite-element method using a geometrically exact element.

The procedure is applicable to shells subjected to a very general set of thermal loading conditions including applied heat flux at the surfaces, convective heat transfer with the surroundings, and equivalent loadings due to specified temperatures on the shell. Two general-purpose computer programs based on the procedure have been developed and are described.

The computer programs have been demonstrated for a variety of shell configurations for which exact solutions were available. These configurations include cylinders under a number of different heating conditions, a conical frustum, a truncated hemisphere, and an annular plate. Results from the computer programs are compared with the exact analytical solutions.

## INTRODUCTION

Current aerospace applications require the structural analyst to have a knowledge of the dynamic response of structural components in a thermal environment. Among the most important of these components are thin shells of revolution. As major components in aeronautical and aerospace vehicles, shells are subjected to thermal inputs such as aerodynamic heating, heating due to contact with an adjacent high-temperature environment, and nonuniform heating such as occurs when a small section of the shell is in contact with an adjacent heated component.

A practical method for computing forced response of shells in a thermal environment is being developed at Langley Research Center. A generally accepted systematic procedure for this problem (outlined below) is being followed:

- (1) Calculate the temperature distribution in the shell due to the thermal inputs.
- (2) Compute the thermal stresses in the shell due to this temperature distribution.

(3) Compute the natural frequencies and mode shapes of the shell in the presence of the thermal stresses.

(4) Calculate the dynamic response of the shell due to applied mechanical loads by modal superposition.

The first and most basic step in the procedure requires a method for computing nonaxisymmetric steady-state temperature distributions in thin shells of revolution having general meridional curvature. The resulting temperature distributions would then be used as input to other computer programs which would proceed through steps (2) to (4). The method used to predict temperature distributions should be fast and accurate, and should be applicable to a wide variety of shell geometries and shells having nonhomogeneous properties.

Two methods have been most successful for solving problems of this type: the finite-difference method (ref. 1), and the finite-element method (refs. 2 to 6). The finite-element method has the following advantages relative to the finite-difference method:

(1) The finite-difference method operates on the governing differential equation whereas the finite-element method operates on an appropriate variational expression that is usually somewhat easier to obtain.

(2) The finite-element method results in a symmetric matrix formulation whereas the matrices in a finite-difference formulation are not always symmetric. Symmetric matrix formulations are desirable because of the body of mathematical tools available for solving the resulting equations. Since computer programs to meet the aforementioned requirements did not exist, a new finite-element procedure and computer program were developed and are described in this paper.

A geometrically exact element was employed because of previous success in applications to vibration analysis. (See ref. 7.) The computer programs have the capability of computing the nonaxisymmetric, steady-state temperature distribution in thin shells of revolution subject to a general set of thermal loading conditions including heat flux and equivalent loads due to temperatures specified on the shell. They have been applied to a variety of shells of revolution. Comparison of results with available exact solutions was made and is presented in this paper.

## SYMBOLS

A            matrix which relates temperatures and derivatives of temperature at ends of element to coefficients of polynomial representing temperature in element

$a_0, a_1, a_2, a_3$	coefficients of polynomial representing temperature distribution in element
C	column matrix used to specify position of a point where temperature is specified and magnitude of specified temperatures (eq. (50))
E	expression which, when minimized, gives governing equations for steady-state temperature distribution in thin shells of revolution
$F_1, F_2$	partitions of thermal force vector corresponding to unknown and known temperatures, respectively
f	thermal force vector
G	expression which, when minimized, gives governing equations for temperature distribution for shell in which temperatures are specified at discrete points
H	convective heat-transfer coefficient
h	thickness of shell
K	number of finite elements used to represent shell
$k(s, z)$	thermal conductivity
$K(s) = \int_{-h/2}^{h/2} k(s, z) dz$	
k	element number
L	length of shell
M	number of circumferences on which temperatures are specified
N	largest Fourier index in a set of calculations
n	Fourier index
P	number of discrete points at which temperature is specified

Q	column matrix whose elements are coefficients of linear terms in E
q	heat flux
R	matrix whose elements are coefficients of a quadratic form in E (eq. (5))
r	perpendicular distance from shell axis to a point on shell middle surface
S	conductivity matrix
$S_{11}, S_{12}, S_{21}, S_{22}$	partitions of S (eq. (40))
s	meridional coordinate
T	specified temperature
U	vector containing temperatures at element junctures
u	temperature
V	column matrix used to specify position of a point where temperature is prescribed and also the magnitude of the temperature (eq. (51))
X	matrix which describes polynomial approximation of temperature distribution in an element (eq. (14))
x	meridional coordinate in an element
y	vector defined in equation (12)
z	coordinate normal to shell middle surface measured from middle surface
$\gamma$	column matrix whose elements are the coefficients of the polynomial approximating the temperature distribution in an element

$$\delta_K^k = \begin{cases} 0 & (k \neq K) \\ 1 & (k = K) \end{cases}$$

$$\delta_1^k = \begin{cases} 0 & (k \neq 1) \\ 1 & (k = 1) \end{cases}$$

$\epsilon$  meridional length of an element

$\theta$  circumferential coordinate

$\lambda$  Lagrange multiplier

Subscripts:

F first edge of shell

I inside surface of shell

k element number or juncture number

L last edge of shell

m index referring to circumference at which temperature is specified

O outside surface of shell

p index which refers to point at which temperature is specified

cn Fourier cosine component

sn Fourier sine component

Superscripts:

cn Fourier cosine component

sn Fourier sine component

T transformed matrix

Primes indicate differentiation with respect to  $x$  or  $s$ .

## ANALYSIS METHOD

In this section of the report, the governing equations are derived. The following procedure is followed: A functional is obtained which, when minimized, yields the

boundary-value problem governing steady-state conduction of heat in a thin shell of revolution. This functional is obtained by specializing a more general expression (refs. 2 to 6) by making the following assumptions:

- (1) There is no time variation of temperature.
- (2) Temperature gradients through the thickness of the shell may be ignored.
- (3) The geometry and properties of the shell are independent of the angular coordinate  $\theta$ .

The functional constitutes the basis for the entire approach. Once it is derived, the analysis follows a well-known procedure described in references 2 to 6.

Each variable in the functional is expanded in a Fourier series around the shell circumference so that the calculation of temperature distribution reduces to computing only the meridional variation of temperature. In accordance with the finite-element method, the shell is represented by a number of elements in which the temperature is approximated by polynomials. In the present procedure the elements are geometrically exact in that no approximation of the shape of the shell is necessary. Conductivity matrices and thermal force vectors are derived for each element and these element matrices are superimposed to form the corresponding matrices for the complete shell. It is assumed in this analysis that both the temperature and its first derivative are continuous at junctures between adjacent elements. According to reference 6, as a minimum, only temperature need be continuous in order to guarantee convergence of the method. Nonetheless, specifying continuity of the first derivative is a reasonable condition to impose since in most of the anticipated applications of the present method, the temperature distributions will be smooth functions. In the cases where the actual temperature distributions have slope discontinuities, the method based on continuous slopes will still converge since the conditions expressed in reference 6 are satisfied. The resulting solution will cause the "corner" at the discontinuity to be rounded.

The matrix equations for determining the temperature distribution are derived by minimizing the approximated functional with respect to the generalized coordinates which in this analysis are the values of temperature and first derivative of temperature at the ends of each element.

For the purposes of the following development, reference is made to figure 1. In this figure  $u(s, \theta)$  represents the temperature of a point on the shell defined by the meridional coordinate  $s$  and the circumferential coordinate  $\theta$ . The ambient temperatures of the media outside of the first and last edges of the shell are represented by  $u_F(\theta)$  and  $u_L(\theta)$ , respectively. The ambient temperature of the medium at the outer surface of the shell is  $u_O(s, \theta)$  and the ambient temperature of the medium at the inner surface of the shell is  $u_I(s, \theta)$ . Heat flux is applied to the outer and inner surfaces of



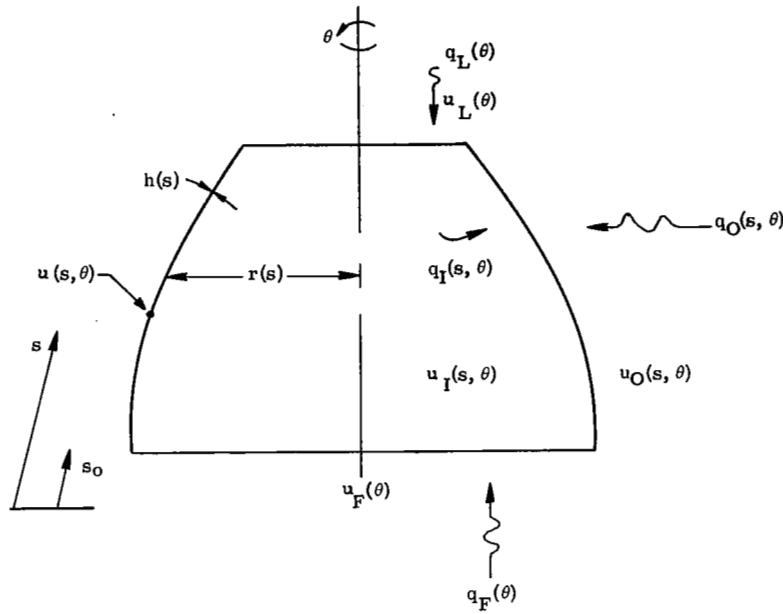


Figure 1.- Nomenclature associated with transfer of heat in a thin shell of revolution.

the shell by sources  $q_O(s, \theta)$  and  $q_I(s, \theta)$ , respectively. The applied heat flux at the first and last edges are given by  $q_F(\theta)$  and  $q_L(\theta)$ , respectively. The perpendicular distance from the axis of revolution to a point on the shell middle surface is given by  $r(s)$ . The distance from the origin of the meridional coordinate  $s$  to the first edge is  $s_0$  and the distance to the last edge is  $L$ . The thickness of the shell is given by  $h(s)$ .

The functional for the temperature distribution for bodies of general shape is well known. (See, for example, refs. 2 to 6.) Specialization of the functional for steady-state heat conduction in isotropic thin shells of revolution gives the following functional:

$$\begin{aligned}
 E = & \frac{1}{2} \iiint k(s, z) \left[ \left( \frac{\partial u}{\partial s} \right)^2 + \frac{1}{r^2(s)} \left( \frac{\partial u}{\partial \theta} \right)^2 \right] r(s) \, ds \, d\theta \, dz + \iint H_O(s) \left( \frac{u^2}{2} - u_O u \right) r(s) \, ds \, d\theta + \iint H_I(s) \left( \frac{u^2}{2} - u_I u \right) r(s) \, ds \, d\theta \\
 & - \iint q_O(s, \theta) u r(s) \, ds \, d\theta - \iint q_I(s, \theta) u r(s) \, ds \, d\theta - \iint_{CF} q_F(\theta) u r(s_0) \, dz \, d\theta - \iint_{CL} q_L(\theta) u r(L) \, dz \, d\theta \\
 & + \iint_{CF} H_F \left( \frac{u^2}{2} - u u_F \right) r(s_0) \, dz \, d\theta + \iint_{CL} H_L \left( \frac{u^2}{2} - u u_L \right) r(L) \, dz \, d\theta
 \end{aligned} \quad (1)$$

where

- (1)  $k(s,z)$  is the thermal conductivity of the shell material
- (2)  $H_O(s)$  and  $H_I(s)$  are the convective heat-transfer coefficients at the outer and inner surfaces of the shell
- (3)  $H_F$  and  $H_L$  are the convective heat-transfer coefficients at the first and last edges of the shell. The integrations indicated by  $\iint_{CF}$  and  $\iint_{CL}$  are taken over the first and last edges, respectively, of the shell.

Radiation effects are entirely neglected in this analysis.

For a shell of revolution, the variables may be represented in terms of their Fourier components as follows:

$$\begin{aligned}
 u(s, \theta) &= \sum_{n=0}^N u_{cn}(s) \cos n\theta + \sum_{n=1}^N u_{sn}(s) \sin n\theta \\
 u_O(s, \theta) &= \sum_{n=0}^N u_{O,cn}(s) \cos n\theta + \sum_{n=1}^N u_{O,sn}(s) \sin n\theta \\
 u_I(s, \theta) &= \sum_{n=0}^N u_{I,cn}(s) \cos n\theta + \sum_{n=1}^N u_{I,sn}(s) \sin n\theta \\
 q_O(s, \theta) &= \sum_{n=0}^N q_{O,cn}(s) \cos n\theta + \sum_{n=1}^N q_{O,sn}(s) \sin n\theta \\
 q_I(s, \theta) &= \sum_{n=0}^N q_{I,cn}(s) \cos n\theta + \sum_{n=1}^N q_{I,sn}(s) \sin n\theta \\
 q_F(\theta) &= \sum_{n=0}^N q_{F,cn} \cos n\theta + \sum_{n=1}^N q_{F,sn} \sin n\theta
 \end{aligned}$$

(Equations continued on next page)

$$\left. \begin{aligned} q_L(\theta) &= \sum_{n=0}^N q_{L,cn} \cos n\theta + \sum_{n=1}^N q_{L,sn} \sin n\theta \\ u_F(\theta) &= \sum_{n=0}^N u_{F,cn} \cos n\theta + \sum_{n=1}^N u_{F,sn} \sin n\theta \\ u_L(\theta) &= \sum_{n=0}^N u_{L,cn} \cos n\theta + \sum_{n=1}^N u_{L,sn} \sin n\theta \end{aligned} \right\} \quad (2)$$

Substitution of equations (2) into equation (1) and integration with respect to  $\theta$  from 0 to  $2\pi$  yields the following form for the functional  $E$ :

$$E = \sum_{n=0}^N E_{cn} + \sum_{n=1}^N E_{sn} \quad (3)$$

The forms of  $E_{cn}$  and  $E_{sn}$  are identical. The formulation of the finite-element representation of  $E_{sn}$  is the same as the formulation for  $E_{cn}$ . Therefore, in order to avoid needless repetition, the derivation will be carried out for  $E_{cn}$  with the understanding that the treatment of  $E_{sn}$  is the same.

The form of  $E_{cn}$  is

$$\begin{aligned} E_{cn} &= \frac{\pi}{2} \int_{-h(s)/2}^{h(s)/2} k(s,z) \left[ (u'_{cn})^2 + \frac{n^2}{r^2} u_{cn}^2 \right] r \, dz \, ds + \pi \int H_O \left( \frac{u_{cn}^2}{2} - u_{O,cn} u_{cn} \right) r \, ds + \pi \int H_I \left( \frac{u_{cn}^2}{2} - u_{I,cn} u_{cn} \right) r \, ds \\ &\quad - \pi \int q_{O,cn} u_{cn} r \, ds - \pi \int q_{I,cn} u_{cn} r \, ds - \pi \int_{-h_F/2}^{h_F/2} q_{F,cn} u_{cn} r(s_0) \, dz - \pi \int_{-h_L/2}^{h_L/2} q_{L,cn} u_{cn} r(L) \, dz \\ &\quad + \pi \int_{-h_F/2}^{h_F/2} H_F \left( \frac{u_{cn}^2}{2} - u_{cn} u_{F,cn} \right) r(s_0) \, dz + \pi \int_{-h_L/2}^{h_L/2} H_L \left( \frac{u_{cn}^2}{2} - u_{cn} u_{L,cn} \right) r(L) \, dz \end{aligned} \quad (4)$$

For  $n = 0$  the value of  $E_{cn}$  is twice the value indicated by equation (4). The integration denoted by  $\int_S$  is taken over the length of the shell meridian and  $h_F$  and  $h_L$  are the shell thicknesses at the first and last edge of the shell, respectively.

## Representation of Shell by Finite-Elements

The shell is represented by geometrically exact elements by which it is meant that the element shape coincides with the part of the shell being represented by the element. A typical idealization is shown in figure 2. By counting elements from the first edge of the shell, the following definitions are made:

$K$	total number of elements
$\epsilon_k$	length of $k$ th element measured along meridian of shell
$x$	meridional coordinate inside $k$ th element, measured along meridian from center of $k$ th element so that $-\frac{\epsilon_k}{2} \leq x \leq \frac{\epsilon_k}{2}$
$s_k$	distance along meridian from first edge of shell to center of $k$ th element
$s_0$	distance from origin of $s$ to first edge of shell

From the foregoing definitions, it follows that

$$s = s_0 + s_k + x$$

Quantities evaluated at the first edge of the  $k$ th element (that is,  $x = -\epsilon_k/2$ ) are denoted by the subscript  $k$ . Quantities evaluated at the second edge of the  $k$ th element (that is,  $x = \epsilon_k/2$ ) are denoted by the subscript  $k + 1$ .

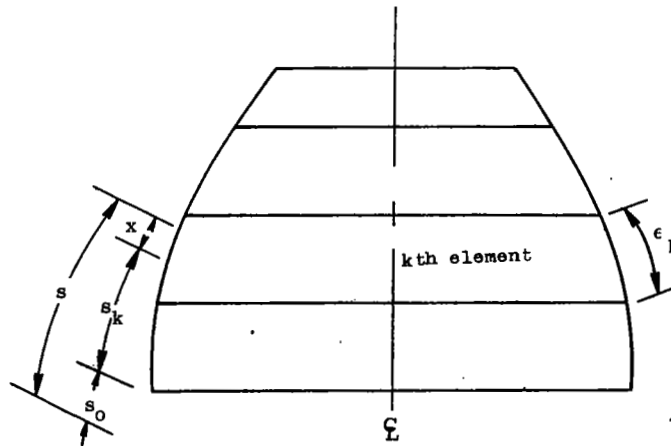


Figure 2.- Typical idealization of shell of revolution by geometrically exact finite elements.

When discussing the complete shell as idealized by finite elements, it is necessary to refer to the term "element junctures." The  $k$ th element juncture is defined to be the circumference which is the first edge of the  $k$ th element. The one exception to this general definition is that the last edge of the shell is designated as the  $K + 1$  element juncture. (See fig. 2.)

By following the conventional steps of the finite-element procedure, the integrations are taken to extend over a single element.

After integration with respect to  $z$ , equation (4) may be written in the following form:

$$\begin{aligned}
 E_{cn,k} = & \frac{\pi}{2} \int_{-\epsilon_k/2}^{\epsilon_k/2} \begin{Bmatrix} u_{cn} \\ u'_{cn} \end{Bmatrix}^T \begin{bmatrix} R_k^{(cn)} \end{bmatrix} \begin{Bmatrix} u_{cn} \\ u'_{cn} \end{Bmatrix} dx - \pi \int_{-\epsilon_k/2}^{\epsilon_k/2} u_{cn} \left[ u_{O,cn}(x) H_O(x) \right. \\
 & + u_{I,cn} H_I(x) + q_{cn}(x) \left. \right] r(x) dx - \left[ \left( \pi h_F q_{F,cn} r_F \right) u_{1,cn} - \frac{1}{2} \left( \pi h_F r_F H_F \right) u_{1,cn}^2 \right. \\
 & + \left( \pi h_F r_F H_F u_{F,cn} \right) u_{1,cn} \left. \right] \delta_1^k - \left[ \left( \pi h_L q_{L,cn} r_L \right) u_{K+1,cn} \right. \\
 & - \frac{1}{2} \left( \pi h_L H_L r_L \right) u_{K+1,cn}^2 + \left( \pi h_L H_L r_L u_{L,cn} \right) u_{K+1,cn} \left. \right] \delta_K^k \quad (5)
 \end{aligned}$$

where  $\begin{bmatrix} R_k^{(cn)} \end{bmatrix}$  is a two-by-two symmetric matrix where elements are functions of  $x$  as given below:

$$\begin{bmatrix} R_k^{(cn)} \end{bmatrix} = \begin{bmatrix} \frac{n^2 K(x)}{r^2(x)} + H_O(x) r(x) + H_I(x) r(x) & 0 \\ 0 & K(x) r(x) \end{bmatrix} \quad (6)$$

$$K(x) = \int_{-h/2}^{h/2} k(x,z) dz \quad (7)$$

$$q_{cn}(x) = q_{O,cn}(x) + q_{I,cn}(x) \quad (8)$$

$r_F = r(s_0) = r$  at first edge of shell

$r_L = r(L) = r$  at last edge of shell

$u_{1,cn}$  temperature component at first edge of shell

$u_{K+1,cn}$  temperature component at last edge of shell

$$\delta_1^k = \begin{cases} 0 & (k \neq 1) \\ 1 & (k = 1) \end{cases}$$

$$\delta_K^k = \begin{cases} 0 & (k \neq K) \\ 1 & (k = K) \end{cases}$$

As an approximation, the temperature is assumed to have the following polynomial form within the  $k$ th element:

$$u_{cn}(x) = a_{0,k}^{(cn)} + a_{1,k}^{(cn)}x + a_{2,k}^{(cn)}x^2 + a_{3,k}^{(cn)}x^3 \quad (9)$$

Equation (9) may be written as

$$u_{cn}(x) = \{ \gamma^{(cn)} \}^T \{ y \} \quad (10)$$

where

$$\{ \gamma^{(cn)} \} = \begin{Bmatrix} a_{0,k}^{(cn)} \\ a_{1,k}^{(cn)} \\ a_{2,k}^{(cn)} \\ a_{3,k}^{(cn)} \end{Bmatrix} \quad (11)$$

$$\{ y \} = \begin{Bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{Bmatrix} \quad (12)$$

The variables  $a_{0,k}^{(cn)}$ ,  $a_{1,k}^{(cn)}$ ,  $a_{2,k}^{(cn)}$ , and  $a_{3,k}^{(cn)}$  are the undetermined coefficients associated with the nth Fourier cosine component in the kth element.

The relationship between the temperature and its first derivative at the ends of an element and the coefficients of the assumed polynomial is derived as follows.

Using equation (9) and a differentiation of both sides of the equation yields

$$\begin{Bmatrix} u_{cn} \\ u'_{cn} \end{Bmatrix} = [X] \{\gamma^{(cn)}\} \quad (13)$$

where

$$[X] = \begin{bmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \end{bmatrix} \quad (14)$$

Inserting  $x = -\epsilon_k/2$  and  $x = \epsilon_k/2$  into appropriate locations in equation (13) and inverting the resulting matrix equation gives the following relation:

$$\{\gamma^{(cn)}\} = [A_k] \{U_k^{(cn)}\} \quad (15)$$

where

$$[A_k] = \begin{bmatrix} 1/2 & \epsilon_k/8 & 1/2 & -\epsilon_k/8 \\ -3/2\epsilon_k & -1/4 & 3/2\epsilon_k & -1/4 \\ 0 & -1/2\epsilon_k & 0 & 1/2\epsilon_k \\ 2/\epsilon_k^3 & 1/\epsilon_k^2 & -2/\epsilon_k^3 & 1/\epsilon_k^2 \end{bmatrix} \quad (16)$$

$$\{U_k^{(cn)}\} = \begin{Bmatrix} u_k^{(cn)} \\ u_k^{(cn)'} \\ u_{k+1}^{(cn)} \\ u_{k+1}^{(cn)'} \end{Bmatrix} \quad (17)$$

Substituting equations (10) and (13) into equation (5) gives  $E_{cn,k}$  in terms of the undetermined coefficients of the polynomials as

$$\begin{aligned}
E_{cn,k} = & \frac{\pi}{2} \int_{-\epsilon_k/2}^{\epsilon_k/2} \{y^{(cn)}\}^T [X]^T [R_k^{(cn)}] [X] \{y^{(cn)}\} dx \\
& - \pi \int_{-\epsilon_k/2}^{\epsilon_k/2} \{y^{(cn)}\}^T \{y\} \left[ u_{O,cn}(x) H_O(x) + u_{I,cn}(x) H_I(x) + q_{cn}(x) \right] dx \\
& - \left[ \pi (h_F q_{F,cn} r_F + h_F r_F H_F u_{F,cn}) u_{1,cn} - \frac{\pi}{2} h_F r_F H_F u_{1,cn}^2 \right] \delta_1^k \\
& - \left[ \pi (h_L q_{L,cn} r_L + h_L H_L r_L u_{L,cn}) u_{K+1,cn} - \frac{\pi}{2} h_L H_L r_L u_{K+1,cn}^2 \right] \delta_K^k \quad (18)
\end{aligned}$$

Define

$$[Z_k^{(cn)}] = \pi \int_{-\epsilon_k/2}^{\epsilon_k/2} [X]^T [R_k^{(cn)}] [X] dx \quad (19)$$

$$\{Q_k^{(cn)}\} = \int_{-\epsilon_k/2}^{\epsilon_k/2} \{y\} \left[ u_{O,cn}(x) H_O(x) + u_{I,cn}(x) H_I(x) + q_{cn}(x) \right] r(x) dx \quad (20)$$

$$B_F^{(cn)} = \pi \left[ h_F r_F (q_{F,cn} + H_F u_{F,cn}) \right] \quad (21)$$

$$b_F = \pi h_F r_F H_F \quad (22)$$

$$B_L^{(cn)} = \pi \left[ h_L r_L (q_{L,cn} + H_L u_{L,cn}) \right] \quad (23)$$

$$b_L = \pi h_L r_L H_F \quad (24)$$



By utilizing equations (19) to (24), equation (18) may be written as

$$E_{cn,k} = \frac{1}{2} \{ \gamma^{(cn)} \}^T \left[ Z_k^{(cn)} \right] \{ \gamma^{(cn)} \} - \{ \gamma^{(cn)} \}^T \left\{ Q_k^{(cn)} \right\} \\ - \left[ B_F^{(cn)} u_{1,cn} - \frac{1}{2} b_F u_{1,cn}^2 \right] \delta_1^k - \left[ B_L^{(cn)} u_{K+1,cn} - \frac{1}{2} b_L u_{K+1,cn}^2 \right] \delta_K^k \quad (25)$$

Substituting equation (15) into equation (25) yields

$$E_{cn,k} = \frac{1}{2} \left\{ U_k^{(cn)} \right\}^T \left[ A_k \right]^T \left[ Z_k^{(cn)} \right] \left[ A_k \right] \left\{ U_k^{(cn)} \right\} - \left\{ U_k^{(cn)} \right\}^T \left[ A_k \right]^T \left\{ Q_k^{(cn)} \right\} \\ - \left[ B_F^{(cn)} u_{1,cn} - \frac{1}{2} b_F u_{1,cn}^2 \right] \delta_1^k - \left[ B_L^{(cn)} u_{K+1,cn} - \frac{1}{2} b_L u_{K+1,cn}^2 \right] \delta_K^k \quad (26)$$

The conductivity matrix and thermal force vector may be identified for each element as  $\left[ S_k^{(cn)} \right]$  and  $\left\{ f_k^{(cn)} \right\}$ , respectively, where

$$\left[ S_k^{(cn)} \right] = \left[ A_k \right]^T \left[ Z_k^{(cn)} \right] \left[ A_k \right] + \left[ b_1 \right] \delta_1^k + \left[ b_K \right] \delta_K^k \quad (27)$$

$$\left\{ f_k^{(cn)} \right\} = \left[ A_k \right]^T \left\{ Q_k^{(cn)} \right\} + \left\{ B_1^{(cn)} \right\} \delta_1^k + \left\{ B_K^{(cn)} \right\} \delta_K^k \quad (28)$$

where

$$\left[ b_1 \right] = \begin{bmatrix} b_F & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (29)$$

$$\left\{ B_1^{(cn)} \right\} = \begin{Bmatrix} B_F^{(cn)} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (30)$$

$$[b_K] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_L & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (31)$$

$$\left\{ B_K^{(cn)} \right\} = \begin{Bmatrix} 0 \\ 0 \\ B_L^{(cn)} \\ 0 \end{Bmatrix} \quad (32)$$

In view of the numbering convention adopted for the elements, the second edge of the  $k$ th element coincides with the first edge of the  $k + 1$  element. In this analysis it is assumed that the following conditions of compatibility hold at each such juncture:

$$\left\{ \begin{matrix} u_{k+1}^{(cn)} \\ u_{k+1}^{(cn)'} \end{matrix} \right\}_{\substack{k \\ \text{element}}} = \left\{ \begin{matrix} u_{k+1}^{(cn)} \\ u_{k+1}^{(cn)'} \end{matrix} \right\}_{\substack{k+1 \\ \text{element}}} \quad (1 \leq k \leq K) \quad (33)$$

These compatibility conditions require that the temperature be continuous with a continuous first derivative at all points along the shell. The equality between the derivatives is strictly valid only for cases where the heat flux and thermal conductivity are continuous.

The functional  $E_{cn}$  associated with the  $n$ th harmonic for the entire shell is given by the following summation:

$$E_{cn} = \sum_{k=1}^K E_{cn,k} \quad (34)$$

where  $E_{cn,k}$  is given by equation (26).

Use of equation (33) in equation (34) gives

$$E_{cn} = \frac{1}{2} \{U^{(cn)}\}^T [S^{(cn)}] \{U^{(cn)}\} - \{U^{(cn)}\}^T \{f^{(cn)}\} \quad (35)$$

where

$S^{(cn)}$  thermal conductivity matrix which is a symmetric positive definite matrix of order  $2(K+1) \times 2(K+1)$  (see ref. 2)

$f^{(cn)}$  thermal force vector which is of order  $2(K+1) \times 1$

$U^{(cn)}$  a vector containing all temperature components and derivatives

Thus,

$$\{U^{(cn)}\} = \left\{ \begin{array}{c} U_1^{(cn)} \\ U_2^{(cn)} \\ \vdots \\ U_K^{(cn)} \end{array} \right\}_{2(K+1) \times 1} \quad (36)$$

In the present procedure, the matrices  $S^{(cn)}$  and  $f^{(cn)}$  are constructed by the well-known procedure of superposition of element matrices as illustrated in figure 3. The superposition to form  $S^{(cn)}$  consists of placing the element matrices together so that the matrix elements in the lower  $2 \times 2$  block of the  $k$ th matrix add to the corresponding matrix elements in the upper  $2 \times 2$  block of the  $k+1$  matrix. In the case of  $f^{(cn)}$ , the superposition consists of placing the element vectors together so that the last two numbers in the  $k$ th vector add to the first two numbers in the  $k+1$  vector.

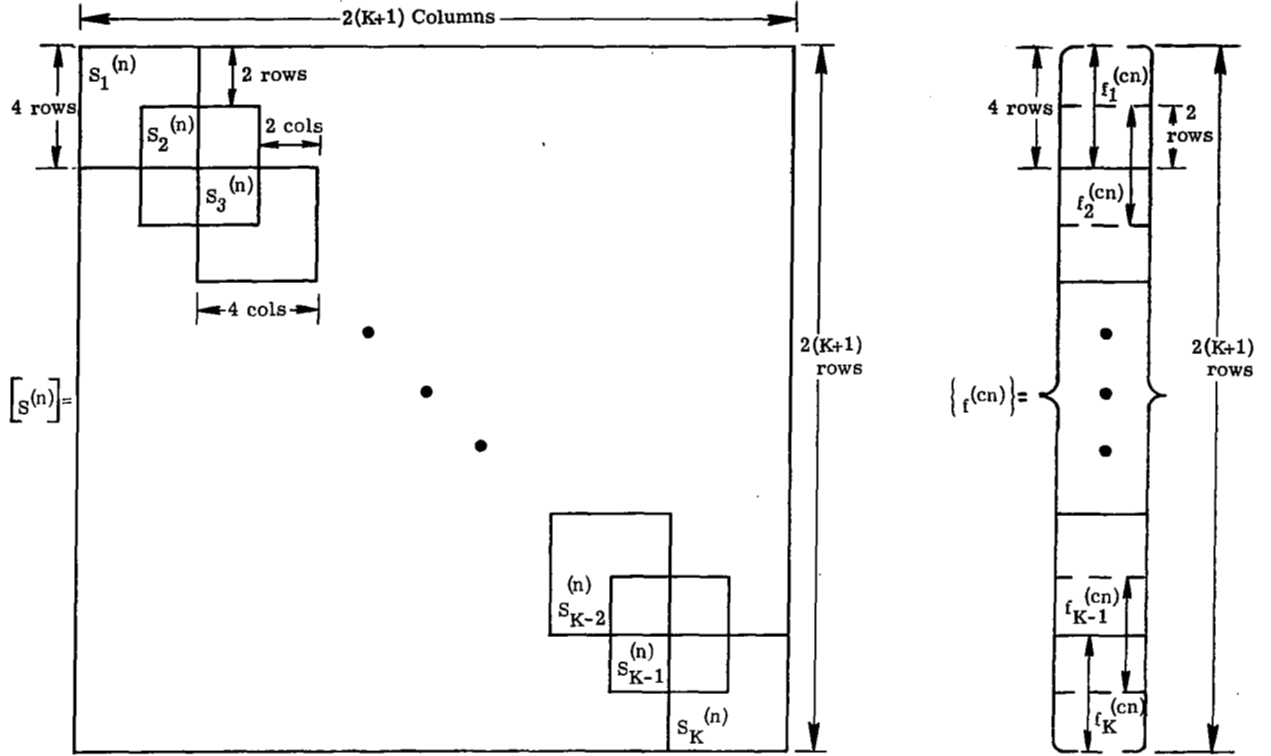


Figure 3.- Illustration of superposition of element matrices.

By recalling that the treatment of  $E_{sn}$  is identical to the treatment of  $E_{cn}$ , equation (35) is substituted into equation (3) and the following expression is obtained:

$$\begin{aligned}
 E = & \sum_{n=0}^N \left( \frac{1}{2} \{U^{(cn)}\}^T [S^{(n)}] \{U^{(cn)}\} - \{U^{(cn)}\}^T \{f^{(cn)}\} \right) \\
 & + \sum_{n=1}^N \left( \frac{1}{2} \{U^{(sn)}\}^T [S^{(n)}] \{U^{(sn)}\} - \{U^{(sn)}\}^T \{f^{(sn)}\} \right)
 \end{aligned} \quad (37)$$

Since  $[S^{(cn)}]$  is identical to  $[S^{(sn)}]$ , they are both denoted by  $[S^{(n)}]$ .

#### Formulation of Governing Matrix Equations

The equations for determining the vectors  $U^{(cn)}$  and  $U^{(sn)}$  are derived by minimizing  $E$  with respect to the variables in these vectors. It is seen from equation (37) that there are no cross products between sine and cosine vectors. Further, there are no cross products between different sine components or between different cosine components. Therefore, the minimization of  $E$  with respect to a given sine or

cosine component gives an equation which involves only that component. Thus, each cosine component and each sine component may be calculated independently. This property is of practical significance and its exploitation will be illustrated in the discussion of the computational method. In the following derivations, the operations for the typical cosine component are performed with the understanding that all the cosine components as well as all the sine components are treated in an identical fashion.

Formulation for case where temperature over entire shell is unknown.- If the temperature at every point on the shell is unknown, the equations for determining a given Fourier component of temperature are found by minimizing  $E$  with respect to each of the elements in the vector  $U^{(cn)}$ . The minimization is equivalent to the following set of equations,  $2(K + 1)$  in number:

$$\left. \begin{aligned} \frac{\partial E}{\partial u_k^{(cn)}} &= 0 & (k = 1, 2, \dots, K + 1) \\ \frac{\partial E}{\partial u_k^{(cn)'}} &= 0 & (k = 1, 2, \dots, K + 1) \end{aligned} \right\} \quad (38)$$

Equations (38) can be expressed in the form:

$$[S^{(n)}] \{U^{(cn)}\} = \{f^{(cn)}\} \quad (39)$$

Equation (39) may be solved for  $\{U^{(cn)}\}$  by inverting the matrix  $[S^{(n)}]$  since it is positive definite and therefore nonsingular (ref. 2).

Recall that the elements in the vector  $\{U^{(cn)}\}$  are values of the Fourier components of temperature and meridional derivatives thereof at each element juncture. By using equation (15), the coefficients of the polynomial within each element may be computed from the elements in  $\{U^{(cn)}\}$ . Then the temperature component within each element may be computed at as many points as desired by use of equation (9).

Formulation for case where temperature is specified on circumferences.- There is no loss in generality in assuming that the circumferences along which temperature is specified are element junctures. When this assumption is made, the vector  $U^{(cn)}$  may be partitioned into an unknown part  $v^{(cn)}$  and known part  $w^{(cn)}$ . Then

$$E_{cn} = \frac{1}{2} \left\{ \begin{matrix} v^{(cn)} \\ w^{(cn)} \end{matrix} \right\}^T \left[ \begin{array}{c|c} S_{11}^{(cn)} & S_{12}^{(cn)} \\ \hline S_{21}^{(cn)} & S_{22}^{(cn)} \end{array} \right] \left\{ \begin{matrix} v^{(cn)} \\ w^{(cn)} \end{matrix} \right\} - \left\{ \begin{matrix} v^{(cn)} \\ w^{(cn)} \end{matrix} \right\}^T \left\{ \begin{matrix} F_1^{(cn)} \\ F_2^{(cn)} \end{matrix} \right\} \quad (40)$$

where

$S_{11}^{(cn)}$  a square matrix whose rows and columns correspond to the unknown elements of  $U^{(cn)}$ . It is of order  $2(K+1) - M$

$S_{12}^{(cn)}$  a rectangular matrix having  $2(K+1) - M$  rows and  $M$  columns. The rows correspond to unknown elements of  $U^{(cn)}$  and the columns correspond to known elements of  $U^{(cn)}$ .

$S_{21}^{(cn)}$  transpose of  $S_{12}^{(cn)}$

$S_{22}^{(cn)}$  a square matrix of order  $M$  whose rows and columns correspond to known elements of  $U^{(cn)}$

$F_1^{(cn)}$  a partition of  $f^{(cn)}$  whose rows correspond to the unknown elements of  $U^{(cn)}$

$F_2^{(cn)}$  a partition of  $f^{(cn)}$  whose rows and columns correspond to the known elements of  $U^{(cn)}$

If the temperature of the shell is specified on the edges of the shell, then the variables  $u_{1,cn}$  and  $u_{k+1,cn}$  are included in the vector  $w^{(cn)}$ . The treatment of edge conditions is discussed in more detail in a later section. Minimizing  $E$  with respect to the elements in  $v^{(cn)}$  yields the equations for determining the unknown temperatures. The minimization is the same as that of equation (38) except the differentiations are taken only with respect to the elements of  $v^{(cn)}$ . The resulting matrix equation may then be written as

$$\left[ S_{11}^{(cn)} \right] \{ v^{(cn)} \} = \left\{ F_1^{(cn)} \right\} - \left[ S_{12}^{(cn)} \right] \{ w^{(cn)} \} \quad (41)$$

This equation is solved by inverting the matrix  $\left[ S_{11}^{(cn)} \right]$ . This matrix is nonsingular because it is a submatrix of the positive definite matrix  $S^{(n)}$ .

Formulation of equations for case where temperatures are prescribed at discrete points.- For the purpose of the following development, the reader is referred to figure 4. The  $p$ th point at which temperature is specified is located by the coordinates  $s_p, \theta_p$ . The temperature at that point is denoted  $T_p$ . The number of the element in which the  $p$ th point is located is denoted by  $k_p$  and the corresponding value of  $s_k$  given by  $s_{kp}$ . The meridional distance from the center of the  $k_p$ th element to the  $p$ th point is denoted by  $x_p$ .

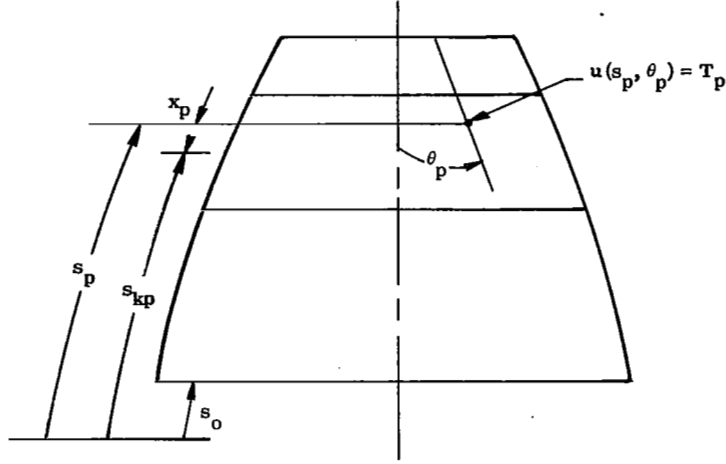


Figure 4.- Nomenclature associated with specification of temperature at a discrete point on a shell.

From these definitions, the following relations are obtained:

$$u(s_p, \theta_p) = T_p \quad (p = 1, 2, \dots, P) \quad (42)$$

$$x_p = s_p - s_{kp} - s_o \quad (p = 1, 2, \dots, P) \quad (43)$$

where  $P$  is the number of discrete points at which temperature is specified.

The derivation of the governing equations for this class of problems is facilitated by the use of Lagrange multipliers. It is required to find stationary values of  $E$  (eq. (3)) subject to the constants of equation (42).

It is therefore necessary to find stationary values of the functional  $G$  where

$$G = E + \lambda_1 [u(s_1, \theta_1) - T_1] + \dots + \lambda_p [u(s_p, \theta_p) - T_p] \quad (44)$$

and  $E$  is given by equation (37). The quantities  $\lambda_1, \dots, \lambda_p$  are Lagrange multipliers. Using the first of equations (2) with equations (10) and (42) gives the following relation:

$$u(s_p, \theta_p) = \sum_{n=0}^N \{\gamma^{(cn)}\}^T \{y_p\} \cos n\theta_p + \sum_{n=1}^N \{\gamma^{(sn)}\}^T \{y_p\} \sin n\theta_p \quad (45)$$

where

$$\{y_p\} = \begin{Bmatrix} 1 \\ x_p \\ 2 \\ x_p \\ 3 \\ x_p \end{Bmatrix} \quad (46)$$

Substituting equation (15) into equation (45) gives

$$u(s_p, \theta_p) = \sum_{n=0}^N \left\{ U_{kp}^{(cn)} \right\}^T \{ \beta_{kp} \} + \sum_{n=1}^N \left\{ U_{kp}^{(sn)} \right\} \{ \xi_{kp} \} \quad (47)$$

where

$$\{ \beta_{kp} \} = [A_{kp}]^T \{ y_p \} \cos n \theta_p \quad (48)$$

$$\{ \xi_{kp} \} = [A_{kp}]^T \{ y_p \} \sin n \theta_p \quad (49)$$

Two vectors  $\{ C_{kp}^{(cn)} \}$  and  $\{ V_{kp}^{(cn)} \}$  are defined as follows:

$$\{ C_{kp}^{(n)} \} = \begin{Bmatrix} 0 \\ \vdots \\ \beta_{kp} \\ \vdots \\ 0 \end{Bmatrix} \quad \begin{array}{c} \overline{\uparrow} \\ 2(kp - 1) \text{ rows} \\ \downarrow \\ \overline{\downarrow} \\ \frac{4}{4} \text{ rows} \\ \uparrow \end{array} \quad (50)$$

$$\{ V_{kp}^{(n)} \} = \begin{Bmatrix} 0 \\ \vdots \\ \xi_{kp} \\ \vdots \\ 0 \end{Bmatrix} \quad \begin{array}{c} \overline{\uparrow} \\ 2(kp - 1) \text{ rows} \\ \downarrow \\ \overline{\downarrow} \\ \frac{4}{4} \text{ rows} \\ \uparrow \end{array} \quad (51)$$



Equation (47) may be written with the use of equations (48) to (51) as

$$u(s_p, \theta_p) = \sum_{n=0}^N \{U^{(cn)}\}^T \{C_{kp}^{(n)}\} + \sum \{U^{(sn)}\} \{V_{kp}^{(n)}\} \quad (p = 1, \dots, P) \quad (52)$$

Use of equation (52) in equation (44) gives required form of  $G$ :

$$\begin{aligned} G = & E + \lambda_1 \sum_{n=0}^N \{U^{(cn)}\}^T \{C_{k1}^{(n)}\} + \lambda_1 \sum_{n=1}^N \{U^{(sn)}\}^T \{V_{k1}^{(n)}\} - \lambda_1 T_1 \\ & + \dots + \lambda_P \sum_{n=0}^N \{U^{(cn)}\}^T \{C_{kP}^{(n)}\} + \lambda_P \sum \{U^{(sn)}\}^T \{V_{kP}^{(n)}\} - \lambda_P T_P \end{aligned} \quad (53)$$

The governing matrix equations for this problem are derived by minimizing  $G$  with respect to the elements in  $U^{(cn)}$  and  $U^{(sn)}$  and with respect to the Lagrange multipliers  $\lambda_1$  to  $\lambda_P$ . In contrast to the problem considered previously, all the harmonics must be computed simultaneously. The minimization of  $G$  implies the following conditions:

$$\left. \begin{aligned} \left. \begin{aligned} \frac{\partial G}{\partial u_k^{(cn)}} &= 0 \\ \frac{\partial G}{\partial u_k^{(cn)'}} &= 0 \end{aligned} \right\} & \begin{aligned} &(n = 0, 1, \dots, N) \\ &(k = 1, 2, \dots, K + 1) \end{aligned} \\ \\ \left. \begin{aligned} \frac{\partial G}{\partial u_k^{(sn)}} &= 0 \\ \frac{\partial G}{\partial u_k^{(sn)'}} &= 0 \end{aligned} \right\} & \begin{aligned} &(n = 1, 2, \dots, N) \\ &(k = 1, 2, \dots, K + 1) \end{aligned} \\ \\ \frac{\partial G}{\partial \lambda_p} &= 0 & (p = 1, 2, \dots, P) \end{aligned} \right\} \quad (54)$$

The number of equations represented by equations (54) is  $4(K + 1)(N + 1/2) + P$ . These equations may be written as follows:

$$\left[ S^{(n)} \right] \{ U^{(cn)} \} - \{ f^{(cn)} \} + \lambda_1 \{ C_{k1}^{(n)} \} + \dots + \lambda_P \{ C_{kP}^{(n)} \} = 0$$

(n = 0, 1, \dots, N) (55a)

$$\left[ S^{(n)} \right] \{ U^{(sn)} \} - \{ f^{(sn)} \} + \lambda_1 \{ V_{k1}^{(n)} \} + \dots + \lambda_P \{ V_{kP}^{(n)} \} = 0$$

(n = 1, 2, \dots, N) (55b)

$$\sum_{n=0}^N \{ C_{kp}^{(n)} \}^T \{ U^{(cn)} \} + \sum_{n=1}^N \{ V_{kp}^{(n)} \}^T \{ U^{(sn)} \} - T_p = 0$$

(p = 1, 2, \dots, P) (55c)

The actual form of this set of equations written as a single matrix equation is

$$\begin{bmatrix} \left[ S^{(0)} \right] & 0 & 0 & \dots & 0 & 0 & \{ C_{k1}^{(0)} \} & \dots & \{ C_{kP}^{(0)} \} \\ 0 & \left[ S^{(1)} \right] & 0 & \dots & 0 & 0 & \{ C_{k1}^{(1)} \} & \dots & \{ C_{kP}^{(1)} \} \\ 0 & 0 & \left[ S^{(1)} \right] & \dots & 0 & 0 & \{ V_{k1}^{(1)} \} & \dots & \{ V_{kP}^{(1)} \} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \left[ S^{(N)} \right] & 0 & \{ C_{k1}^{(N)} \} & \dots & \{ C_{kP}^{(N)} \} \\ 0 & 0 & 0 & \dots & 0 & \left[ S^{(N)} \right] & \{ V_{k1}^{(N)} \} & \dots & \{ V_{kP}^{(N)} \} \\ \{ C_{k1}^{(0)} \}^T & \{ C_{k1}^{(1)} \}^T & \{ S_{k1}^{(1)} \}^T & \dots & \{ C_{k1}^{(N)} \} & \{ V_{k1}^{(N)} \} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \{ C_{kP}^{(0)} \}^T & \{ C_{kP}^{(1)} \}^T & \{ V_{kP}^{(1)} \}^T & \dots & \{ C_{kP}^{(N)} \}^T & \{ V_{kP}^{(N)} \}^T & 0 & \dots & 0 \end{bmatrix} \begin{Bmatrix} \{ U^{(c0)} \} \\ \{ U^{(c1)} \} \\ \{ U^{(s1)} \} \\ \vdots \\ \{ U^{(cN)} \} \\ \{ U^{(sN)} \} \\ \lambda_1 \\ \vdots \\ \lambda_P \end{Bmatrix} = \begin{Bmatrix} \{ f^{(c0)} \} \\ \{ f^{(c1)} \} \\ \{ f^{(s1)} \} \\ \vdots \\ \{ f^{(cN)} \} \\ \{ f^{(sN)} \} \\ T_1 \\ \vdots \\ T_P \end{Bmatrix} \quad (56)$$

The matrix in equation (56) is nonsingular (ref. 2) so that the solution of the equation may be effected by inverting the matrix. The resulting vector can be broken down into the subvectors  $U^{(cn)}$  and  $U^{(sn)}$  for each value of  $n$ . Coefficients of the polynomials may be determined by use of equation (13) and the polynomial representations of  $U_{cn}(x)$  and  $U_{sn}(x)$  within each element may be computed.

If the temperatures, in addition to being specified at discrete points, are also specified along circumferences, the only changes in equation (55) are as follows:

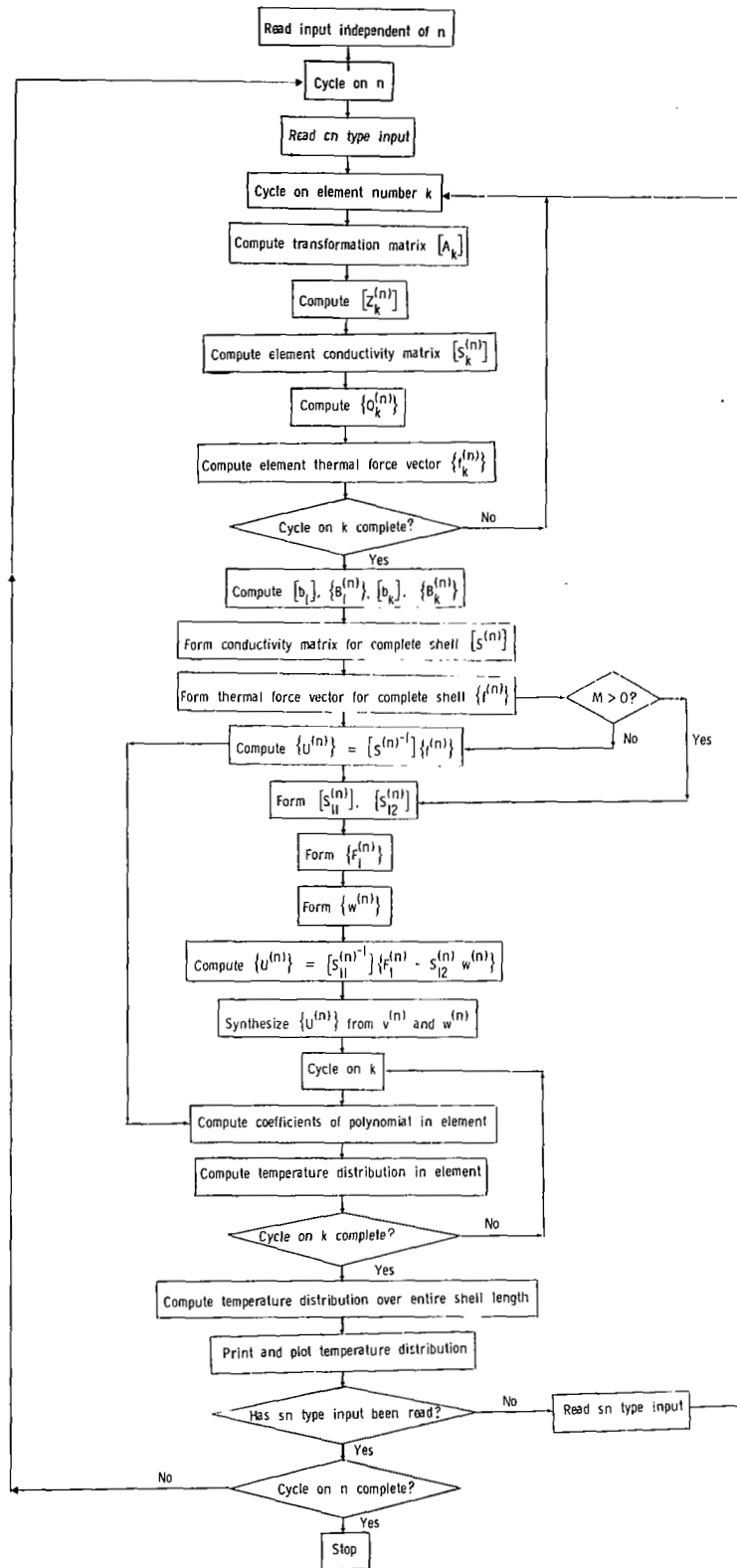
$$\begin{aligned} [S^{(n)}] & \text{ is replaced by } [S_{11}^{(n)}] \\ \{f^{(cn)}\} & \text{ is replaced by } \{F_1^{(cn)}\} - [S_{12}^{(n)}] \{w^{(cn)}\} \\ \{f^{(sn)}\} & \text{ is replaced by } \{F_1^{(sn)}\} - [S_{12}^{(n)}] \{w^{(sn)}\} \\ \{U^{(cn)}\} & \text{ is replaced by } \{v^{(cn)}\} \\ \{U^{(sn)}\} & \text{ is replaced by } \{v^{(sn)}\} \end{aligned}$$

## OUTLINE OF COMPUTER PROGRAMS

In this section of the paper, the basic computing procedure is illustrated by means of flow charts. In addition, the input to the computer programs is described along with the form of the output. The section is divided into two parts. The first part deals with the computer program which treats the case where temperatures, when specified on the shell, are specified only on circumferences and the second deals with the program for the case where temperatures are specified at discrete points as well as on circumferences.

### Computing Method for Calculations Wherein Known Temperatures are Specified on Complete Circumferences Only

The organization of the computing steps for this type of calculation is illustrated in block diagram 1.



The inputs are of three types: input independent of the circumferential harmonic, cosine-type input, and "sine type" input. The input which is independent of the circumferential harmonic is as follows:

$K$	number of elements used to represent shell
$\epsilon_k$	length of kth element where $k = 1, 2, \dots, K$
$s_0$	meridional distance from origin of $s$ to first edge of shell
$r(s)$	shell radius
$h(s)$	shell thickness
$K(s)$	integral of thermal conductivity through shell thickness (see eq. (7))
$H_I(s)$	convective heat-transfer coefficient at inner surface of shell
$H_O(s)$	convective heat-transfer coefficient at outer surface of shell
NBEG	initial value of $n$
NLAST	final value of $n$
$M$	number of element junctures at which temperature is specified
$m(i)$	an array in which the $i$ th entry is the number of the $i$ th element juncture at which temperature is specified, $i = 1, 2, \dots, M$
$H_F$	convective heat-transfer coefficient at first edge of shell
$H_L$	convective heat-transfer coefficient at last edge of shell
ININ	number of integration intervals used to carry out the evaluation of matrices $Z$ and $Q$ by the trapezoidal rule
IPLQT	number of locations within each element that temperature distribution is printed

The integrations required for computing matrices  $Z$  and  $Q$  are performed numerically by using the trapezoidal rule with 100 integration intervals. The temperature distributions are computed and printed at 10 locations in each element. Both of these numbers may be changed by varying the input quantities  $ININ$  and  $IPLØT$ .

The cosine-type input is as follows:

$u_{I,cn}(s)$	nth cosine harmonic of temperature of medium in contact with inside of shell
$u_{O,cn}(s)$	nth cosine harmonic of temperature of medium in contact with outside of shell
$q_{cn}(s)$	nth cosine harmonic of net heat flux applied to surface of shell
$q_{F,cn}$	nth cosine harmonic of heat flux applied at first edge of shell
$u_{F,cn}$	nth cosine harmonic of temperature of medium in contact with first edge of shell
$q_{L,cn}$	nth cosine harmonic of heat flux applied at second edge of shell
$u_{L,cn}$	nth cosine harmonic of temperature of medium in contact with second edge of shell
$T_{m,cn}$	nth cosine harmonic of specified temperature at element juncture $m(i)$ (where $i = 1, 2, \dots, M$ )

The sine-type input is as follows:

$u_{I,sn}(s)$	nth sine harmonic of temperature of medium in contact with inside of shell
$u_{O,sn}(s)$	nth sine harmonic of temperature of medium in contact with outside of shell
$q_{sn}(s)$	nth sine harmonic of net heat flux applied to surface of shell
$q_{F,sn}$	nth sine harmonic of heat flux applied at first edge of shell
$u_{F,sn}$	nth sine harmonic of temperature of medium in contact with first edge of shell
$q_{L,sn}$	nth sine harmonic of heat flux applied at last edge of shell

$u_{L,sn}$  nth sine harmonic of temperature of medium in contact with last edge of shell

$T_{m,sn}$  nth sine harmonic of specified temperature at element juncture  $m(i)$   
(where  $i = 1, 2, \dots, M$ )

Edge conditions must be specified at each edge of the shell. These edge conditions can be any of the following types:

- (1) specified temperature at edge
- (2) applied heat flux at edge
- (3) convective edge condition
- (4) insulated edge condition

The appropriate edge condition at an edge is specified by use of the input quantities  $H_F$ ,  $H_L$ ,  $q_{F,cn}$ ,  $q_{F,sn}$ ,  $q_{L,cn}$ ,  $q_{L,sn}$ ,  $u_{F,cn}$ ,  $u_{F,sn}$ ,  $u_{L,cn}$ , and  $u_{L,sn}$ . The method of specifying edge conditions is summarized in table I.

TABLE I.- METHOD OF SPECIFYING EDGE CONDITIONS

Edge condition	Method of specifying edge condition
Specified temperature at first edge	$H_F = 0$ $q_{F,cn} = q_{F,sn} = u_{F,cn} = u_{F,sn} = 0$ $T_{1,cn}$ and $T_{1,sn}$ specified
Specified temperature at last edge	$H_L = 0$ $q_{L,cn} = q_{L,sn} = u_{L,cn} = u_{L,sn} = 0$ $T_{K+1,cn}$ and $T_{K+1,sn}$ specified
Applied heat flux at first edge	$q_{F,cn}$ and $q_{F,sn}$ specified $H_F = u_{F,cn} = u_{F,sn} = 0$
Applied heat flux at last edge	$q_{L,cn}$ and $q_{L,sn}$ specified $H_L = u_{L,cn} = u_{L,sn} = 0$
Convective edge condition at first edge	$q_{F,cn} = q_{F,sn} = 0$ $H_F$ specified $u_{F,cn}$ and $u_{F,sn}$ specified
Convective edge condition at last edge	$q_{L,cn} = q_{L,sn} = 0$ $H_L$ , $u_{L,cn}$ , $u_{L,sn}$ specified
First edge insulated	$q_{F,cn} = q_{F,sn} = 0$ $u_{F,cn} = u_{F,sn}$ $H_F = 0$
Last edge insulated	$q_{L,cn} = q_{L,sn} = 0$ $u_{L,cn} = u_{L,sn} = 0$ $H_L = 0$
Applied heat flux combined with convection at first edge	$q_{F,cn}$ , $q_{F,sn}$ specified $u_{F,cn}$ , $u_{F,sn}$ specified $H_F$ specified
Applied heat flux combined with convection at last edge	$q_{L,cn}$ , $q_{L,sn}$ specified $u_{L,cn}$ , $u_{L,sn}$ specified $H_L$ specified

The output of this computer program consists of tabulations of the functions  $U_{cn}(s)$  and  $U_{sn}(s)$  for all values of  $n$  in the calculations.

Computing Method for Calculations Wherein Known Temperatures  
are Specified at Discrete Points as Well as on  
Complete Circumferences

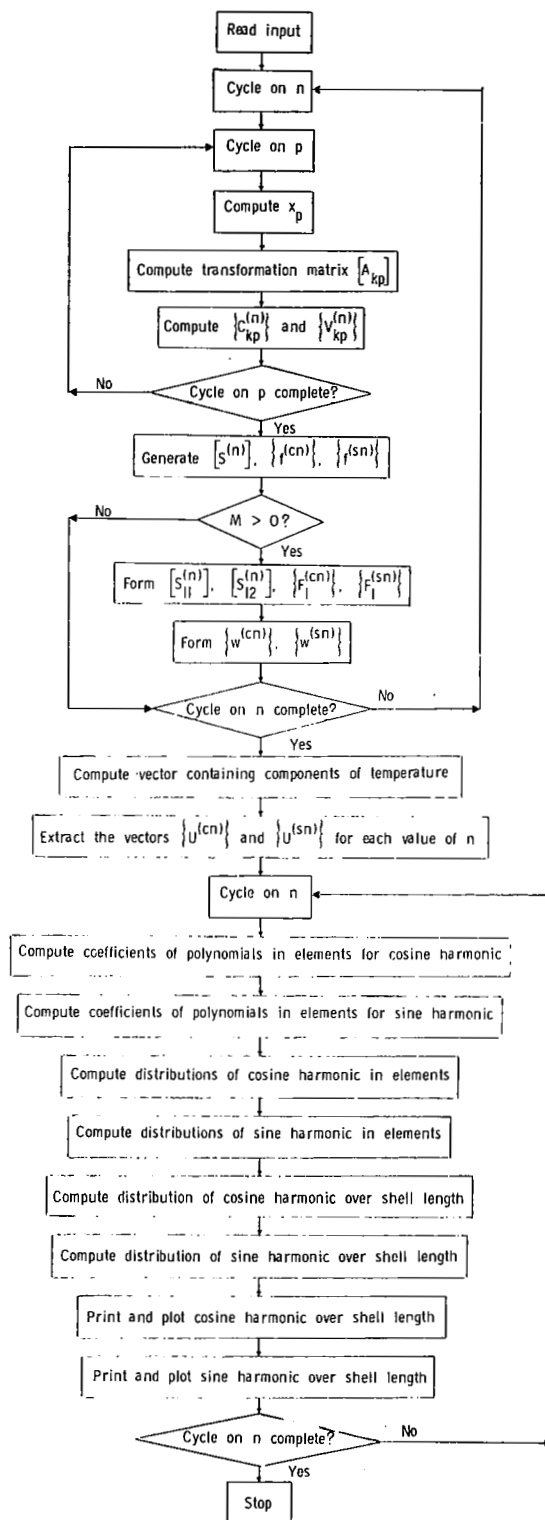
The organization of the computing steps for this type of calculation is illustrated in block diagram 2.

The input to this program includes all the input of the previous program plus the following additional input:

$P$	number of discrete points at which temperature is specified
$s_p$	meridional coordinate of $p$ th point where temperature is specified ( $p = 1, 2, \dots, P$ )
$\theta_p$	circumferential coordinate of $p$ th point where temperature is specified ( $p = 1, 2, \dots, P$ )
$T_p$	temperature specified at $p$ th point ( $p = 1, 2, \dots, P$ )
$k_p$	element number in which $p$ th point is located ( $p = 1, 2, \dots, P$ )

The output of this program consists of tabulations of the functions  $U_{cn}(s)$  and  $U_{sn}(s)$  for the range of  $n$  considered.





## APPLICATION OF METHOD

### Description of Shells Analyzed

In order to demonstrate the versatility of the present method and the accuracy of the results obtained, a number of sample calculations were made. Seven shell configurations were analyzed. For six of these, exact solutions were used for comparison with results of the present method. These solutions were either available from previous investigators or are derived in the appendix. The seventh configuration was of interest but no exact solution was available. The shell configurations were as follows:

- (1) An annular disk insulated at the outer edge, held at fixed temperature at inner edge with convection from the lateral faces. Results are compared with the exact solution from reference 8.
- (2) A cylindrical shell with both edges at fixed temperature with convection to a medium at a uniform ambient temperature. Results are compared with an exact solution derived in the appendix.
- (3) A conical frustum subjected a nonaxisymmetric heat flux over the outer surface. Edges are held at fixed temperatures. Results are compared with an exact solution derived in the appendix.
- (4) A truncated hemisphere subjected to a uniform heat flux at its outer surface. Edges are kept at fixed temperatures. Results are compared with an exact solution derived in the appendix.
- (5) A cylindrical shell with the circumference at midspan of the cylinder held at a fixed temperature and edges held at a different fixed temperature. Results are compared with the exact solution in reference 9.
- (6) A cylindrical shell with a generator held at a fixed temperature. Edges of the shell are insulated. Results are compared with the exact solution in reference 9.
- (7) A cylindrical shell with a single point held at fixed temperature. Edges are insulated.

Dimensions of the shells and numerical values of the parameters describing the thermal environment are assigned convenient values to simplify the calculations. Parameters describing the shells analyzed are listed in table II. The corresponding input to the computer programs are listed in table III. As indicated in table III, configuration 5 was analyzed by using two different finite-element representations, one which had 10 equally spaced elements and another which had 20 elements with more elements near the center of the cylinder than near the ends.

TABLE II.- SAMPLE CALCULATIONS

Parameter	Annular plate $n = 0$	Cylinder: heated edges $n = 0$	Conical frustum $0 \leq n \leq 5$	Truncated hemisphere $n = 0$	Cylinder: heated midlength $n = 0$	Cylinder: heated generator $0 \leq n \leq 4$	Cylinder: heated point $0 \leq n \leq 4$
$h(s)$	0.001	0.001	0.001	0.001	0.001	0.001	0.001
$K(s)$	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$r(s)$	$s$	1.0	$\sqrt{3/2}s$	$\cos(s)$	1.0	1.0	1.0
$r'(s)$	1.0	0.0	$\sqrt{3/2}$	$-\sin(s)$	0.0	0.0	0.0
$H_O(s)$	0.5	1.0	0.0	0.0	1.0	1.0	1.0
$H_I(s)$	0.5	0.0	0.0	0.0	0.0	0.0	0.0
$u_O(s, \theta)$	0.0	1.0	0.0	0.0	1.0	0.0	0.0
$u_I(s, \theta)$	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$q_O(s, \theta)$	0.0	0.0	$\cos n\theta$	1.0	0.0	0.0	0.0
$q_I(s, \theta)$	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Condition at first edge	$u = 1$	$u = 0$	$u = 1$	$u = 1$	$u = 1$	$u' = 0$	$u' = 0$
Condition at last edge	$u' = 0$	$u = 0$	$u = 6$	$u = 2$	$u = 1$	$u' = 0$	$u' = 0$
Additional condition	None	None	None	None	$u(0.5) = 0$	$u = 1$ at $\theta = 0$	$u(0.5, 0) = 1.0$
Meridional length, $L$	1.0	1.0	1.0	1.0	1.0	1.0	1.0

TABLE III.- INPUT TO PROGRAM FOR SAMPLE CALCULATIONS

Parameter	Annular plate	Cylinder: heated edges	Conical frustum	Truncated hemisphere	Cylinder: heated midlength, 10 elements	Cylinder: heated midlength, 20 elements	Cylinder: heated generator	Cylinder: heated point
$K$	5	5	5	5	10	20	5	10
$\epsilon_k$	$a_{0.1}$	$a_{0.2}$	$a_{9/5}$	$a_{0.2}$	$a_{0.1}$	(b)	$a_{0.2}$	$a_{1/6}$
$s_o$	0.5	0	1.0	0.0	0.0	0.0	0.0	0.0
$r(s)$	$s$	1.0	$(\sqrt{3/2})s$	$\cos(s)$	1.0	1.0	1.0	1.0
$h(s)$	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
$K(s)$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$H_I(s)$	0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$H_O(s)$	0.5	1.0	0.0	0.0	1.0	1.0	1.0	1.0
NBEG	0	0	0	0	0	0	0	0
NLAST	0	0	0	0	0	0	4	4
$M$	1	2	2	2	2	2	0	0
$m(i)$	1	1, 6	1, 6	1, 6	1, 6, 11	1, 11, 21	----	----
$H_F$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$H_L$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$u_{I,cn}(s)$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$u_{O,cn}(s)$	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
$q_{cn}(s)$	0.0	0.0	1.0	1.0	0.0	0.0	0.0	0.0
$q_{F,cn}$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$u_{F,cn}$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$q_{L,cn}$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$u_{L,cn}$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$T_{m,cn}$	1.0	0.0, 0.0	1.0, 6.0	1.0, 2.0	1.0, 10.0, 1.0	1.0, 10.0, 1.0	0.0	0.0
$u_{I,sn}(s)$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$u_{O,sn}(s)$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$q_{sn}(s)$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$q_{F,sn}$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$u_{F,sn}$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$q_{L,sn}$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$u_{L,sn}$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$T_{m,sn}$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$P$	----	----	----	----	----	----	11	1
$k_p$	----	----	----	----	----	----	(c)	4
$s_p$	----	----	----	----	----	----	(d)	0.5
$\theta_p$	----	----	----	----	----	----	$a_{0.0}$	0.0
$T_p$	----	----	----	----	----	----	$a_{1.0}$	1.0

<sup>a</sup>All equal.<sup>b</sup> $\epsilon_1$  to  $\epsilon_5 = 0.08$ ,  $\epsilon_6$  to  $\epsilon_{15} = 0.02$ ,  $\epsilon_{16}$  to  $\epsilon_{20} = 0.08$ .<sup>c</sup> $k_1 = 1$   $k_2 = 1$   $k_3 = 2$   $k_4 = 2$   $k_5 = 3$   $k_6 = 3$   $k_7 = 4$   $k_8 = 4$   $k_9 = 5$   $k_{10} = 5$   $k_{11} = 5$ .<sup>d</sup> $s_1 = 0$   $s_2 = 0.1$   $s_3 = 0.2$   $s_4 = 0.3$   $s_5 = 0.4$   $s_6 = 0.5$   $s_7 = 0.6$   $s_8 = 0.7$   $s_9 = 0.8$   $s_{10} = 0.9$   $s_{11} = 1.0$ .

## RESULTS AND DISCUSSION

The results of the sample calculations are presented in figures 5 to 13. In figures 5 to 10, the predicted temperature distributions from the present analysis are compared with exact solutions from the appendix (eqs. (A4) to (A12)). The curves in figures 5 to 10 which represent the exact solutions were obtained by use of computer programs which evaluate equations (A4) to (A12) at a large number of points. Figures 11 to 13 contain only results from the present method since no exact solution was available for comparison.

The agreement between results of the present method and the exact solutions for the first four configurations (figs. 5 to 8) is excellent. The numerical values were in agreement to eight significant figures; thus, the corresponding curves are coincident.

For the fifth configuration (fig. 9) using an idealization of 10 equal elements, the present results and the exact solution are coincident except in the vicinity of the center of the cylinder where the present solution deviates from the exact solution. This slight error is attributed to the inability of the present solution to predict the slope discontinuity exhibited by the exact solution at the center of the cylinder. It will be recalled that one of the assumptions made in the present formulation was that the temperature distributions have continuous derivatives.

In order to show that errors associated with smoothing of slope discontinuities may be made negligible, a second calculation was made. In this calculation 20 elements were used, smaller elements being placed in the vicinity of the center of the cylinder than at the ends. The result of this calculation was a temperature distribution which was coincident with the exact solution for plotting purposes. (See fig. 9.) In that figure the 20 element calculation and exact solution are both represented by the solid line.

The results for configuration 6 are presented in table IV and figure 10. The results were obtained as follows: Using ranges of  $n$  of zero to two, zero to three, and zero to four, the computer program of block diagram 2 computed the functions  $u_{cn}(s)$  and  $u_{sn}(s)$ . All the  $u_{sn}(s)$  functions were found to be identically zero and thus agreed with the exact solution in equation (A12). The  $u_{cn}(s)$  functions were constants and are tabulated in table IV along with the exact Fourier components from the appendix.

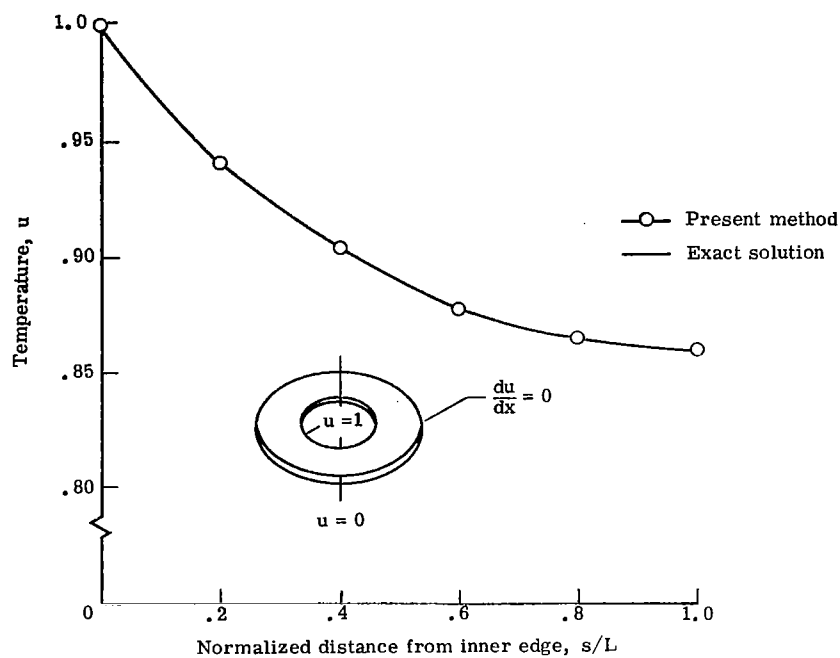


Figure 5.- Temperature in a disk computed by present method and exact solution.

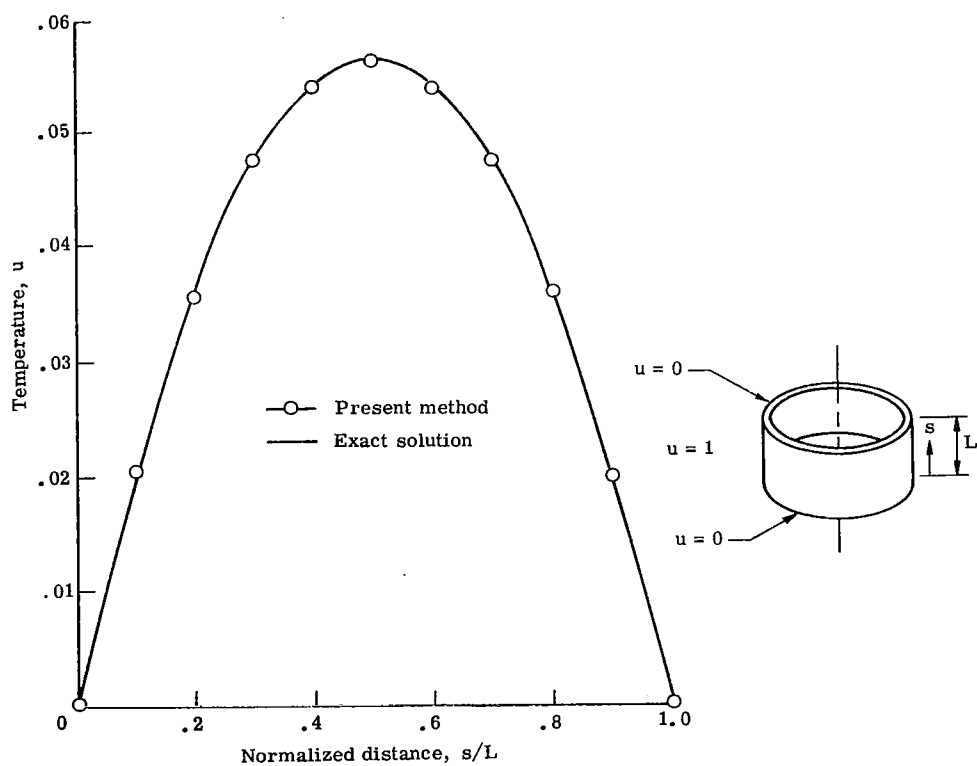


Figure 6.- Temperature in a cylinder with convection to unit ambient temperature and ends at zero.

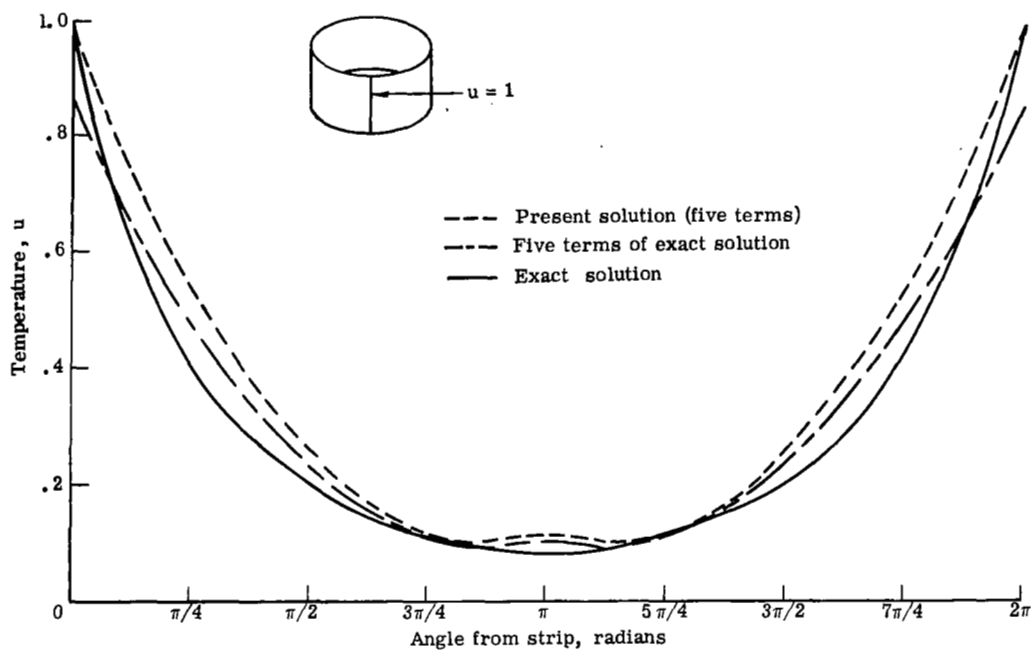


Figure 10.- Temperature distribution in a cylindrical shell with an axial strip held at constant temperature.

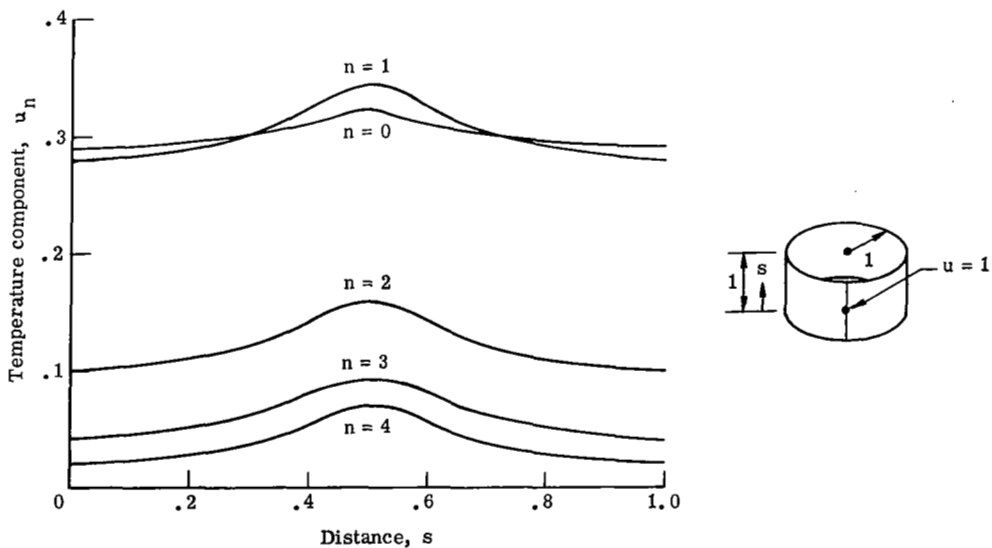


Figure 11.- Fourier components of temperature distribution in a cylinder with specified temperature at a single point.

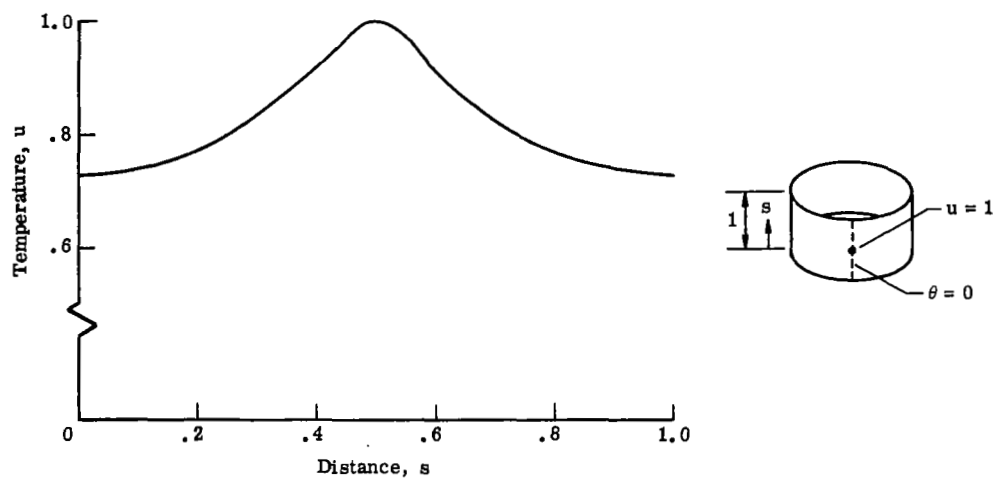


Figure 12.- Temperature profile along  $\theta = 0$  for specified temperature at a single point on a cylinder.

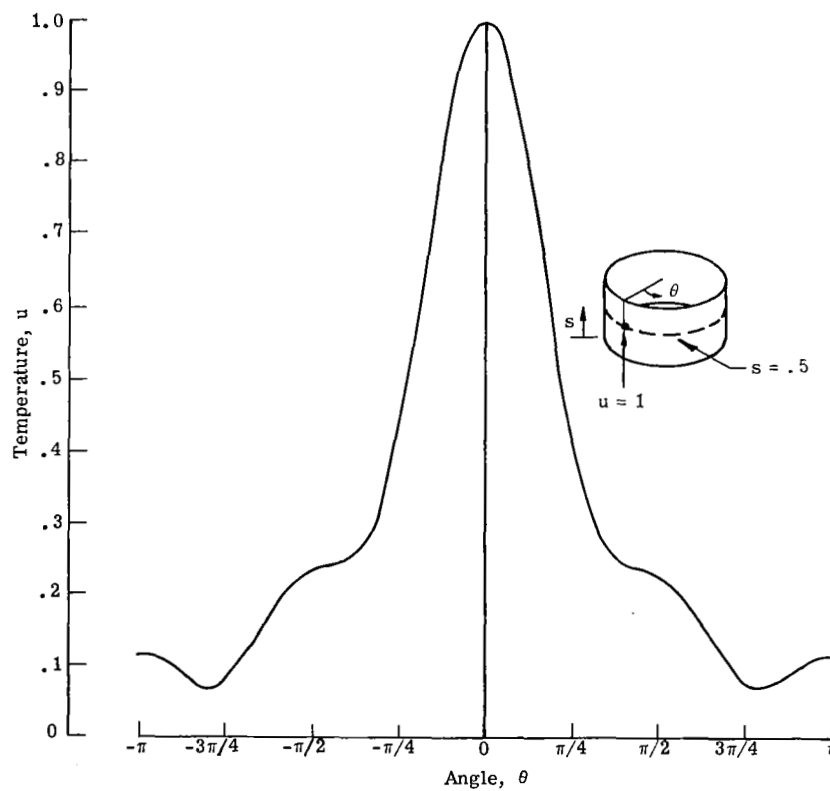


Figure 13.- Temperature profile along  $s = 0.5$  for specified temperature at a single point on a cylinder.

TABLE IV.- FOURIER COSINE HARMONICS OF TEMPERATURE FOR  
CYLINDER WITH GENERATOR HELD AT UNIT TEMPERATURE

Fourier harmonic	Present analysis (3 terms)	Present analysis (4 terms)	Present analysis (5 terms)	Exact solution (eq. (A12))
n = 0	0.417	0.385	0.368	0.318
n = 1	.417	.385	.368	.318
n = 2	.167	.153	.147	.127
n = 3	----	.0769	.0736	.0636
n = 4	----	-----	.0433	.0374

Inspection of table IV shows that each Fourier harmonic from even the best calculation is higher than the corresponding exact harmonic by 16 percent. This error is much larger than had been encountered on any of the previous calculations because approximations had to be made in this problem which limit the accuracy. These approximations are as follows:

(1) The condition that the entire generator is held at the prescribed temperature is approximated by the condition that a finite number of discrete points are held at the prescribed temperature.

(2) In contrast to the previous calculations, all the Fourier harmonics must be computed simultaneously. (See eq. (56).) The storage limitations of the computer require a truncation of the number of harmonics which may be computed in a given calculation. This truncation necessarily limits the accuracy of the computed harmonics because theoretically the exact solution would be achieved only when an infinite number of harmonics were computed simultaneously.

As an additional aid to evaluating the results of this calculation, the Fourier series representations of the temperature were computed for two sets of coefficients in table IV, the coefficients from the present five-term calculation and the exact coefficients. These representations are plotted in figure 10 along with the exact (converged) solution. In figure 10 the short-dashed line is the five-term representation based on the present analysis. The short-long-dashed line is the five-term representation using exact coefficients. The solid line is the converged solution of equation (A10). The hump in the vicinity of  $\theta = \pi$  in both five-term solutions is characteristic of a truncated Fourier series.

It can be seen from figure 10 that the result from the present analysis is in reasonably good agreement with the two other results. The present solution is an upper bound to both of the other solutions. The maximum disparity between the present five-term



solution and the exact five-term representation is 16 percent at  $\theta = 0$ . The maximum disparity between the present five-term solution and the converged solution is 29 percent at  $\theta = \pi/4$ . The maximum difference between the exact five-term representation and the converged solution is 16 percent at  $\theta = 0$ .

The final sample calculation is the temperature distribution in a cylinder wherein a single point is held at unit temperature. The results are shown in figures 11 to 13. This problem was chosen to demonstrate the applicability of the present method to a problem for which an exact solution was not available and hence a numerical solution was necessary. The point on the cylinder at which temperature is specified is taken to be at the midlength of the cylinder at  $\theta = 0$ . The temperature distribution would then be expected to be symmetric about the line  $s = L/2$  and the generator  $\theta = 0$ .

The functions  $u_{cn}(s)$  and  $u_{sn}(s)$  were computed for a range of  $n$  of zero to four. All the  $u_{sn}(s)$  functions were identically zero; thus, the symmetry of the temperature about  $\theta = 0$  is verified. In addition, as shown in figure 11, the  $u_{cn}(s)$  functions are symmetric about  $s = L/2$ .

In order to illustrate further the nature of the temperature distribution, two profiles through the heated point are calculated. The temperature profile along the generator  $\theta = 0$  is shown in figure 12. This curve was calculated by adding the curves of figure 11. The profile along the circumference  $s = 0.5$  is shown in figure 13. This curve was constructed from the equation

$$u(0.5, \theta) = \sum_{n=0}^4 u_{cn}(0.5) \cos n\theta \quad (57)$$

The appearance of the curve in the vicinity of  $\theta = \pi/2$  and  $\theta = -\pi/2$  is characteristic of a truncated Fourier series. As the number of Fourier terms is increased, the irregularities in the curve should disappear.

The computer programs are very efficient as evidenced by the fact that the total running time required on the Control Data 6600 computer system to perform all the calculations in the paper was less than 2 minutes.

### CONCLUDING REMARKS

An analytical procedure based on the finite-element method has been developed for computing steady-state temperature distributions in thin shells of revolution. The shells may have general meridional curvature and nonhomogeneous thermal properties. The procedure is applicable to shells subjected to a very general set of thermal loading conditions including applied heat flux at the surfaces, convective heat transfer to the

surroundings, and equivalent loadings due to specified temperatures on the shell. General descriptions of two computer programs based on the procedure are given. The first program deals with problems wherein temperatures are specified on complete circumferences only. The second deals with problems wherein temperatures are specified both on circumferences and at discrete points.

The computer programs have been applied to a number of configurations for which exact analytical solutions were available:

- (1) An annular plate with specified temperature at the inner radius and insulated at the outer radius
- (2) A cylindrical shell with edges at fixed temperatures and with convection to surroundings
- (3) A conical frustum subjected to a nonuniform heat flux over the outer surface
- (4) A truncated hemisphere subjected to uniform heat flux at the outer surface
- (5) A cylindrical shell with circumference at midlength held at constant temperature
- (6) A cylindrical shell with a generator held at constant temperature

In addition to these configurations, a seventh configuration was analyzed for which no other solution was available. This configuration was a cylindrical shell in which a single point was held at a prescribed temperature. In general, the temperature distributions predicted by the present procedure were in excellent agreement with the exact solutions. Inaccuracies occurred in the solution for configurations 6 and 7 in which it was theoretically necessary to solve for all the Fourier components of temperature simultaneously. Thus, a truncation in the size of the set of equations solved was necessary.

In problems wherein the exact temperature distribution has a slope discontinuity such as configuration 5, the present method predicts a continuous slope. In this case by using a refined representation of the shell in the neighborhood of the discontinuity, the error associated with this phenomenon was made negligible.

Langley Research Center,  
National Aeronautics and Space Administration,  
Hampton, Va., December 28, 1970.

## APPENDIX

### EXACT SOLUTIONS FOR TEMPERATURE DISTRIBUTIONS

Taking the first variation of the functional of equation (1) and setting it equal zero in the usual manner of the calculus of variations gives the required boundary-value problem:

$$\frac{\partial}{\partial s} \left( K \frac{\partial u}{\partial s} \right) + K \frac{r'}{r} \frac{\partial u}{\partial s} + \frac{K}{r^2} \frac{\partial^2 u}{\partial \theta^2} - (H_O + H_I)u + H_O u_O + H_I u_I + q_O + q_I = 0 \quad (A1)$$

$$\left. \begin{array}{l} \text{or} \\ u \text{ prescribed} \end{array} \right\} \begin{array}{l} K \frac{\partial u}{\partial s} + h_F q_F - h_F H_F (u - u_F) = 0 \\ (s = s_O) \end{array} \quad (A2)$$

$$\left. \begin{array}{l} \text{or} \\ u \text{ prescribed} \end{array} \right\} \begin{array}{l} K \frac{\partial u}{\partial s} - h_L q_L + h_L H_L (u - u_L) = 0 \\ (s = L) \end{array} \quad (A3)$$

Solutions to equations (A1) to (A3) are obtained for the first six shell configurations.

#### Annular Disk

The terms in equation (A1) appropriate to an annular disk with constant thermal conductivity subjected to convection to uniform surroundings at zero temperature are

$$r = s$$

$$r' = 1$$

$$\frac{\partial}{\partial \theta} = 0$$

$$H_O = H_I = H = \text{Constant}$$

$$u_O = u_I = 0$$

## APPENDIX - Continued

$$q_O = q_I = 0$$

$$K = \text{Constant}$$

Equation (A1) becomes

$$\frac{d^2 u}{ds^2} + \frac{1}{s} \frac{du}{ds} - 2 \frac{H}{K} u = 0$$

The boundary conditions are taken to be

$$u\left(\frac{1}{2}\right) = T_0$$

$$\left. \frac{du}{ds} \right|_{s=1} = 0$$

The solution to the preceding boundary-value problem is given in reference 8 as

$$u(s) = T_0 \frac{I_0(\lambda s) K_1(\lambda) + K_0(\lambda s) I_1(\lambda)}{I_0\left(\frac{\lambda}{2}\right) K_1(\lambda) + K_0\left(\frac{\lambda}{2}\right) I_1(\lambda)} \quad (\text{A4})$$

where

$I_0$       modified Bessel function of the first kind of order zero

$I_1$       modified Bessel function of the first kind of order one

$K_0$       modified Bessel function of the second kind of order zero

$K_1$       modified Bessel function of the second kind of order one

$$\lambda^2 = 2 \frac{H}{K}$$

Numerical results were obtained for the following values of the parameters:

$$H = \frac{1}{2}$$

$$K = T_0 = 1$$

## APPENDIX - Continued

### Cylindrical Shell With Convection to Constant Ambient Temperature

For the cylindrical shell with convection to constant ambient temperature, when constant properties of the shell and surroundings are assumed, the terms in equation (A1) are

$$K = \text{Constant}$$

$$r = \text{Constant}$$

$$H_O = \text{Constant}$$

$$H_I = 0$$

$$q_O = q_I = 0$$

$$\frac{\partial}{\partial \theta} = 0$$

$$u_O = \text{Constant}$$

$$u_I = 0$$

The governing differential equation is reduced to

$$K \frac{d^2 u}{ds^2} - H_O u + H_O u_O = 0$$

The boundary conditions are taken to be

$$u(0) = u(L) = 0$$

The solution is

$$u(s) = u_O (1 - \cosh \lambda s) + u_O \left( \frac{\cosh \lambda L - 1}{\sinh \lambda L} \right) \sinh \lambda s \quad (\text{A5})$$

where

$$\lambda^2 = \frac{H_O}{K}$$

Equation (A5) was evaluated for the following values of the parameters:

$$u_O = H_O = K = L = 1$$

### Conical Frustum Subjected to Nonaxisymmetric Heat Flux at Outer Surface

For the conical frustum subjected to nonaxisymmetric heat flux at the outer surface, the terms in equations (A1) to (A3) are

# APPENDIX - Continued

$$K = \text{Constant}$$

$$r(s) = \frac{\sqrt{3}}{2}s$$

$$r' = \frac{\sqrt{3}}{2}$$

$$H_O = H_I = 0$$

$$q_O = Q(\theta)$$

$$q_I = 0$$

$$u(s_O, \theta) = T_O(\theta)$$

$$u(L, \theta) = T_L(\theta)$$

Equations (A1) to (A3) are

$$\frac{\partial^2 u}{\partial s^2} + \frac{1}{s} \frac{\partial u}{\partial s} + \frac{4}{3s^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{Q(\theta)}{K} = 0$$

$$u(s_O, \theta) = T_O(\theta)$$

$$u(L, \theta) = T_L(\theta)$$

Consider the case:

$$Q(\theta) = Q_n \cos n\theta$$

$$T_O(\theta) = T_{O,n} \cos n\theta$$

$$T_L(\theta) = T_{L,n} \cos n\theta$$

It follows that

$$u(s, \theta) = u_n(s) \cos n\theta$$

thus,

$$\frac{d^2 u_n}{ds^2} + \frac{1}{s} \frac{du_n}{ds} - \frac{4n^2}{3s^2} u_n + \frac{Q_n}{K} = 0$$

$$u_n(s_O) = T_{O,n}$$

$$u_n(L) = T_{L,n}$$

# APPENDIX - Continued

The solution for  $n \neq 0$  is

$$u_n(s) = A_n s^{\mu_n} + B_n s^{-\mu_n} + \frac{Q_n s^2}{K(\mu_n^2 - 4)} \quad (\text{A6})$$

where

$$\mu_n = \frac{2}{\sqrt{3}} n$$

$$A_n = \frac{L^{-\mu_n} \left[ T_{0,n} - \frac{Q_n s_0^2}{K(\mu_n^2 - 4)} \right] - s_0^{-\mu_n} \left[ T_{1,n} - \frac{Q_n L^2}{K(\mu_n^2 - 4)} \right]}{s_0^{\mu_n} L^{-\mu_n} - s_0^{-\mu_n} L^{\mu_n}}$$

$$B_n = \frac{-L^{-\mu_n} \left[ T_{0,n} - \frac{Q_n s_0^2}{K(\mu_n^2 - 4)} \right] + s_0^{\mu_n} \left[ T_{1,n} - \frac{Q_n L^2}{K(\mu_n^2 - 4)} \right]}{s_0^{\mu_n} L^{-\mu_n} - s_0^{-\mu_n} L^{\mu_n}}$$

For  $n = 0$ , equations (A1) to (A3) become

$$\frac{d^2 u_0}{ds^2} + \frac{1}{s} \frac{du_0}{ds} + \frac{Q_0}{K} = 0$$

$$u_0(s_0) = T_0$$

$$u_0(L) = T_L$$

The solution is

$$u_0(s) = A_0 + B_0 \log_e s - \frac{Q_0 s^2}{4K} \quad (\text{A7})$$

where

$$A_0 = \frac{\log_e L \left( T_0 + \frac{Q_0 s_0^2}{4K} \right) - \log_e s_0 \left( T_L + \frac{Q_0 L^2}{4K} \right)}{\log_e (L/s_0)}$$

# APPENDIX - Continued

$$B_0 = \frac{T_L + \frac{Q_0 L^2}{4K} - T_0 - \frac{Q_0 s_0^2}{4K}}{\log_e(L/s_0)}$$

Numerical results were obtained for the following values of the parameters:

$$s_0 = 1$$

$$L = 10$$

$$Q_n = 1$$

$$\left. \begin{array}{l} T_{0,n} = 1 \\ T_{1,n} = 6 \end{array} \right\} \quad (n = 0, 1, \dots, 5)$$

$$K = 1$$

## Truncated Sphere Subjected to Axisymmetric Heat Flux at Outer Surface

For the truncated sphere subjected to axisymmetric heat flux at the outer surface, the appropriate terms in equations (A1) to (A3) are

$$K = \text{Constant}$$

$$r = R \cos(s/R)$$

$$r' = -\sin(s/R)$$

$$q_I = 0$$

$$q_O = Q = \text{Constant}$$

$$\frac{\partial}{\partial \theta} = 0$$

$$u(0) = T_0$$

$$u(L) = T_L$$

where  $R$  is the radius of the sphere. Equation (A1) takes the form



## APPENDIX - Continued

$$(1 - \eta^2) \frac{d^2 u}{d\eta^2} - 2\eta \frac{du}{d\eta} + \frac{QR^2}{K} = 0$$

where

$$\eta = \sin \frac{s}{R}$$

The general solution with  $\eta$  as the independent variable is

$$u(\eta) = \frac{A}{2} \log \frac{1 + \eta}{1 - \eta} + B + \frac{QR^2}{2K} \log(1 - \eta^2)$$

The solution with  $s$  as the independent variable is

$$u(s) = \frac{A}{2} \log \left[ \frac{1 + \sin(s/R)}{1 - \sin(s/R)} \right] + B + \frac{QR^2}{K} \log \left[ \cos\left(\frac{s}{R}\right) \right] \quad (A8)$$

where

$$A = \frac{2 \left\{ T_L - T_0 - \frac{QR^2}{K} \log \left[ \cos\left(\frac{s}{R}\right) \right] \right\}}{\log \left[ \frac{1 + \sin(L/R)}{1 - \sin(L/R)} \right]}$$

$$B = T_0$$

Numerical results were obtained by evaluating equation (A8) for the following values of the parameters:

$$R = 1$$

$$L = 1$$

$$T_0 = 1$$

$$T_L = 2$$

$$Q = 1$$

### Cylinder With Circumference at Midlength Held at Fixed Temperature

The solution for the cylinder with circumference at midlength held at fixed temperature is given in reference 9 as

# APPENDIX - Continued

$$u(s) = \begin{cases} V_1 + \frac{V_0 - V_1}{\sinh\left(\frac{\mu L}{2}\right)} \sinh \mu s & \left(0 \leq s \leq \frac{L}{2}\right) \\ V_1 + \frac{V_0 - V_1}{\sinh\left(\frac{\mu L}{2}\right)} \sinh \mu (L - s) & \left(\frac{L}{2} \leq s \leq L\right) \end{cases} \quad (A9)$$

where

$$\mu^2 = \frac{H_O}{K}$$

$V_1$  temperature of edges of cylinder

$V_0$  temperature of circumference at midspan of cylinder

$L$  length of cylinder

Numerical results were obtained by evaluating equation (A9) for the following values of the parameters:

$$H_O = K = 1$$

$$V_1 = 1$$

$$V_0 = 10$$

$$L = 1$$

## Cylinder With a Generator Held at Fixed Temperature

The exact solution for the cylinder with a generator held at fixed temperature is given in reference 9. The solution is:

$$u(\theta) = T \frac{\cosh \mu(\theta - \pi)}{\cosh \mu\pi} \quad (A10)$$

where

$$\mu^2 = \frac{H_O}{K}$$

$T$  temperature of generator

## APPENDIX - Concluded

For comparing this solution with results of the present analysis, the Fourier series components of the expression in equation (A10) are computed.

Thus,

$$u(\theta) = \sum_{n=0}^N a_n \cos n\theta + \sum b_n \sin n\theta \quad (\text{A11})$$

where

$$\left. \begin{aligned} a_0 &= \frac{T}{\mu\pi} \tanh \mu\pi \\ a_n &= \frac{2T}{\pi(1+n^2)} \tanh \mu\pi \\ b_n &= 0 \end{aligned} \right\} \quad (\text{A12})$$

For the numerical calculations,

$$T = H_O = K = 1$$

thus,

$$\mu = 1$$

## REFERENCES

1. Dusenberre, George Merrick: Heat-Transfer Calculations by Finite Differences. Int. Textbook Co., c.1961.
2. Becker, E. B.; and Parr, C. H.: Application of the Finite Element Method to Heat Conduction in Solids. Technical Rep. No. S-117 (Contracts DA-01-021 AMC-11536(Z) and DAAH01-67-C-0947), Rohm and Haas Co., Nov. 1967.
3. Wilson, Edward L.; and Nickell, Robert E.: Application of the Finite Element Method to Heat Conduction Analysis. Nucl. Eng. Des., vol. 4, no. 3, Oct. 1966, pp. 276-286.
4. Brocci, R. A.: Analysis of Axisymmetric Linear Heat Conduction Problems by Finite Element Method. Paper No. 69-WA/HT-37, Amer. Soc. Mech. Eng., 1969.
5. Emery, A. F.; and Carson, W. W.: Evaluation of Use of the Finite Element Method in Computation of Temperature. Paper No. 69-WA/HT-38, Amer. Soc. Mech. Eng., 1969.
6. Zienkiewicz, O. C.; and Cheung, Y. K.: The Finite Element Method in Structural and Continuum Mechanics. McGraw-Hill Book Co., Inc., c.1967.
7. Adelman, Howard M.; Catherines, Donnell S.; and Walton, William C., Jr.: A Method for Computation of Vibration Modes and Frequencies of Orthotropic Thin Shells of Revolution Having General Meridional Curvature. NASA TN D-4972, 1969.
8. Carslaw, H. S.; and Jaeger, J. C.: Conduction of Heat in Solids. First ed., The Clarendon Press (Oxford), 1947.
9. Ong, Chung-Chun: Effects of Thermal Stresses on Free Vibrations of Thin Cylindrical Shells. Ph. D. Dissertation, Northwestern Univ., Aug. 1967.

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